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Two basic divisions of logic, propositional calculus and predicate calculus, are examined. The concepts of well formed formulae, true well formed formulae and provable well formed formulae are presented. Lastly the completeness of the two calculi is demonstrated. A brief history of logic precedes the technical discussion.

LOGIC

by

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Logic, like whiskey, loses its beneficial effect when taken in too large quantities.

Lord Dunsany

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INTRODUCTION

This thesis is meant to be an introduction to logic. The first two chapters give general background information. The last two chapters give the details of two of the more basic divisions of logic.

Chapter III is on propositional calculus. First the expressions allowable in this discipline are quite precisely defined. Then it is decided, again in a precise manner, which of these allowable expressions are provable and which are true. Lastly it is demonstrated that an expression of the discipline is true if and only if it is provable.

Chapter IV is on predicate calculus. The same procedure as in Chapter III is used to explore this discipline.

CHAPTER I

LOGIC'S PLACE IN MATHEMATICS

All disciplines in mathematics have a similar construction and are called deductive theories. They start with certain primitive or undefined terms that seem immediately understandable. Other terms are explained with the help of the primitive terms or previously explained terms and are called defined terms. Certain statements of the discipline which seem evident are chosen as primitive statements or axioms and are assumed to be true. Further statements are assumed to be true, and called theorems, only if they can be derived from the axioms, definitions and other statements whose validity has been previously established. [6]

In all disciplines except logic, the laws of logic, without proof of their validity, are used along with the primitive terms and statements in the proof of the theorems. In some divisions of mathematics logic is not the only discipline whose laws are assumed true. Arithmetic presupposes logic; it is expedient for geometry to presuppose arithmetic and logic. Logic is the one discipline in mathematics that is not based on any other discipline, but is basic to all. [6]

CHAPTER II

A BRIEF HISTORY OF LOGIC

Logic was first considered as a discipline unto itself by Aristotle (384-322) in the fourth century B.C. By describing or stating the laws of logic in ordinary language he formed the basis for what is presently called traditional logic. After Aristotle the study of logic stagnated and it was not until the seventeenth century that mathematical or symbolic logic began to develop. Symbolic logic differs from traditional logic in its use of specially devised marks to symbolize directly the thing in question. This use of symbolism added a clarity, precision and compactness which allowed logic to revive and grow. [5] Leibnitz (1646-1716) was the first serious student of symbolic logic. He envisioned a symbolic language that would analyze all concepts into their ultimate constituents. Leibnitz's work in logic remained largely unpublished in his lifetime and it was not until the nineteenth century that symbolic logic experienced continuous development. [7] George Boole (1815-1864) made the most significant contribution of this era in his presentation of a complete and workable calculus. This was the first purely symbolic system and it was of great historical significance. W. S. Jevons (1835-1882) and Charles S. Pierce (1839-1914) modified Boole's system to the present algebra of logic. [2] It remained for Gottlob Frege (1848-1925) to give a formalized form to the deductive theories.

In 1879 he presented predicate calculus, in its modern form, and the notions of propositional functions and quantifiers. Bertrand Russell (1872-1970) and Alfred North Whitehead (1861-1947) extended and made more rigorous the ideas of Frege in their monumental work Principia Mathematica (1910-1913). [7] In 1930 Gödel (1906-) succeeded in proving that predicate calculus is complete - that it is possible to produce all the logically valid formulas in a mechanical fashion. The next year Gödel proved that arithmetic is incomplete, there being statements in arithmetic that can be neither proved nor disproved. [1]

Logic continues to develope. More work has been done in logic in the last century than in all previous centuries. The idea of computability, generated by the study of logic, has a special significance in this age of computers.

CHAPTER III

PROPOSITIONAL CALCULUS

Propositional calculus is the most elementary branch of logic, one which is basic to others. It deals with propositions (individual and composite), connectives that relate the propositions, and rules of inference. The ultimate goal of this chapter is to show that the propositional calculus is complete. This is done by defining what we mean by true and provable wff and by showing that a wff is true if and only if it is provable. [3]

A. BASICS

The study of propositional calculus begins with a formal language. Each symbol of the language denotes only the symbol itself. This is quite different from the usual use of symbols as names that denote other objects.

Three different types of symbols are needed for propositional calculus. First we need an infinite number of symbols to denote independent propositions - a, b, c, etc. Next we require two connectives - $\sim$ and $\vee$. (Actually one connective would suffice but this would generate a very cumbersome language.) Lastly we need parentheses, (and) , for punctuation. Note that the lower case letters, (,), $\vee$, and $\sim$ are the only formal symbols. Any other symbols, and all upper case letters, are the usual symbols which denote something else.

Next a few definitions are needed.

DEFINITION: An <u>expression</u> is a finite string of our formal symbols.

DEFINITION: A <u>well formed formula</u> (wff) is an expression with one of the following forms:

1. (x), where x is an independent proposition
2. (~A), where A is a wff
3. (A∨B), where A and B are wff.

For example, given that a and b are independent propositions, (a), (~(a)), (b) and ((~(a))   (b)) are wff.

DEFINITION: An <u>atomic wff</u> is a wff in which neither ~ nor ∨ occur.

DEFINITION: A <u>composite wff</u> is a wff which is not atomic.

B. PARENTHESES OMITTING CONVENTIONS

The prime purpose of the parentheses is to indicate the main connective. When the main connective of a wff A is ~, there is a wff B such that A is (~B). When the main connective of a wff A is ∨, there are wff B and C such that A is (B∨C). From the definition of wff it can be seen that each composite wff has exactly one main connective. When parentheses do not help determine the main connective they may be omitted.

We shall use three new symbols - ∧ , → , and ↔ - to abbreviate certain wff. In the following table A and B are wff.

Conventional form	Abbreviated form
((~A)∨B)	A → B
(~((~A)∨(~B)))	A ∧ B
(~((~((~A)∨B))∨(~((~B)∨A))))	A ↔ B also (A → B) ∧ (B → A)

Note that " → ", " ↔ ", and " ∧ " are symbols in the conventional sense, not symbols of the formal language. It is possible however to use a different set of marks for the formal language. For example $\wedge$ and $\sim$ are sometimes used instead of $\vee$ and $\sim$ for the formal symbols. In such a case $\vee$, $\rightarrow$ and $\leftrightarrow$ generally are similarly introduced for notational convenience.

In order to further reduce the need for parentheses the connective symbols are ordered. From weakest to strongest they are $\sim$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$. Thus if $\leftrightarrow$ and $\wedge$ are the only two connectives in a wff, $\leftrightarrow$ is the main connective.

The last parentheses omitting convention is to put a dot or several dots over a symbol to strengthen its bracketing power. The more dots a symbol has, the stronger it is. From weakest to strongest the symbols are $\sim$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\dot{\sim}$, $\dot{\vee}$, $\dot{\wedge}$, $\dot{\rightarrow}$, $\dot{\leftrightarrow}$, $\ddot{\sim}$, $\ddot{\vee}$, etc. Thus the wff $(\sim(A \vee B))$ may be written $\dot{\sim} A \vee B$.

C. TRUE WFF

In propositional calculus to define what it means for a wff to be true we must first define a mapping, G, of the set of all wff onto $\{T,F\}$. Let H be any assignment of all atomic wff into $\{T,F\}$. The mapping G can then be defined as follows:

1. if A is atomic $\quad G(A) = H(A)$

2. if $A = (\sim B)$ $\quad G(A) = \begin{cases} T & \text{if } G(B) = F \\ F & \text{if } G(B) = T \end{cases}$

3. if $A = (B \vee C)$ $\quad G(A) = \begin{cases} F & \text{if } G(B) = G(C) = F \\ T & \text{otherwise.} \end{cases}$

A wff A is said to be <u>true</u> if $G(A) = T$ for every possible assignment, H, of the atomic wff in A. For example consider the true wff

$\sim a \vee (a \vee b)$. As is seen below, for each of the four possible assignments of a and b, $G(\sim a \vee (a \vee b)) = T$.

	H(a)	H(b)	G(~a)	G(a∨b)	G(~a∨(a∨b))
1st assignment	T	T	F	T	T
2nd assignment	T	F	F	T	T
3rd assignment	F	T	T	T	T
4th assignment	F	F	T	F	T

On the other hand it can be seen that $(a \vee b)$ is not a true wff because when $H(a) = H(b) = F$, $G(a \vee b) = F$.

D. INDUCTION PRINCIPLE FOR WFF

We shall be interested in showing wff have certain properties. The principle which follows provides a general method to demonstrate that a wff possesses a certain property.

INDUCTION PRINCIPLE FOR WFF [3]: Each wff has property P provided

1. each atomic wff has property P
2. $(\sim A)$ has property P whenever A has property P
3. $(A \vee B)$ has property P whenever A and B have property P.

DEMONSTRATION: Assume for contradiction that there is a property P such that the assumptions of the principle are satisfied but there is a wff which does not possess the property. Let A be the wff that has the fewest connectives and which doesn't possess the property. From the definition of wff either 1) there is a B such that A is $\sim B$ or 2) there are wff B and C such that A is $B \vee C$. In the first case $(A = \sim B)$ the wff B has one less connective than the wff A. Since A is the wff with the fewest connectives that does not have the property, B, with one less connective must have the property. But by the second assumption in the principle if B has the property then $\sim B$ has the property. Since A is $\sim B$, A must have the property, which is a contradiction. Assuming A is $B \vee C$ leads similarly to a contradiction. Thus the assumption that there is a wff that does not have the property must be wrong and the principle is established.

E. PROVABLE WFF

In order to define provable we must first define quite precisely what a proof is.

DEFINITION: A <u>proof</u> is a finite sequence of wff such that each wff G either has one of the following forms

I. 1. $A \vee A \to A$

2. $A \to A \vee B$

3. $A \vee B \to B \vee A$

4. $(A \to B) \to (C \vee A \to C \vee B)$

or

II. G is preceeded by two wff of the form A and $A \to G$.

The four wff under I are <u>axiom</u> <u>schemes</u> for the propositional calculus (A1, A2, A3 and A4) and II is a rule of inference which is called <u>Modus</u> <u>Ponens</u> (MP).

For example the following sequence of wff is a proof.

1.	$A \vee B \to B \vee A$	[A3]
2.	$A \vee B \to B \vee A \dot{\to} C \vee (A \vee B) \to C \vee (B \vee A)$	[A4 A/A∨B B/B∨A]
3.	$C \vee (A \vee B) \to C \vee (B \vee A)$	[MP 1,2]

The justification for each step is in brackets. The first wff is in the form of the third axiom with no changes. The second wff is in the form of the fourth axiom with the A in A4 replaced by $A \vee B$ and the B in A4 replaced by $B \vee A$. The third wff is justified by Modus Ponens with A replaced by wff 1 and $A \to G$ replaced by wff 2.

DEFINITION: A wff is provable if it is the last term in some proof. "$\vdash A$" means " A is provable".

From the preceeding example of a proof we see that $C \vee (A \vee B) \to C \vee (B \vee A)$ is provable.

THEOREM 1: $\vdash C \vee (A \vee B) \to C \vee (B \vee A)$

DEMONSTRATION: Above

F. INDUCTION PRINCIPLE FOR PROVABLE WFF

We shall be interested in showing that all provable wff are true. The following principle presents a method for demonstrating that a provable wff has a certain property. [3]

INDUCTION PRINCIPLE FOR PROVABLE WFF: Each provable wff has property P provided:

1. each member of the axiom scheme has property P
2. whenever there are wff A and $A \to B$ which are both provable and both possess property P then B possesses property P.

DEMONSTRATION: Assume for contradiction that there is a property P such that the assumptions of the principle hold but there is a provable wff that does not have the property. Let A be such a wff. In the proof of A there must be a first wff B which does not have the property. Since B is a wff in a proof it must either be an example of the axiom scheme or a consequence of MP. Since the assumptions of the principle hold it cannot be an example of the axiom scheme and not have the property. Therefore it must be a consequence of MP, which means it must be preceeded by C and $C \to B$. Both of these wff are provable and since they preceed B, the first wff without the property, they must have property P. But by 2 in the theorem if C and $C \to B$ are provable and have the property then B must have the property. This contradiction establishes the principle.

G. ALL PROVABLE WFF ARE TRUE

This theorem will establish half of the requirements necessary for propositional calculus to be complete. The more difficult part is to prove all true wff are provable.

THEOREM: All provable wff are true.

DEMONSTRATION: By the induction principle for provable wff we need only show that all the axioms are true and that when A and $A \to B$ are true then B is true. The following truth tables establish the truth of the axioms.

A1

G(A)	G(A∨A)	G(A∨A → A)
T	T	T
F	F	T

A2

G(A)	G(B)	G(A∨B)	G(A → A∨B)
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

A3

$G(A)$	$G(B)$	$G(A \vee B)$	$G(B \vee A)$	$G(A \vee B \rightarrow B \vee A)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

A4

$G(A)$	$G(B)$	$G(C)$	$G(A \rightarrow B)$	$G(C \vee A)$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

$G(C \vee B)$	$G(C \vee A \rightarrow C \vee B)$	$G(A \rightarrow B \dot{\rightarrow} C \vee A \rightarrow C \vee B)$
T	T	T
T	T	T
T	T	T
F	F	T
T	T	T
T	T	T
T	T	T
F	T	T

From the following chart it can be seen that when A and $A \rightarrow B$ are true then B is true.

$G(A)$	$G(B)$	$G(\sim A)$	$G(\sim A \vee B) = G(A \rightarrow B)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Thus all provable wff must be true.

H. THEOREMS

The following theorems are all necessary for the final theorem that all true wff are provable.

THEOREM 1: $\vdash C \vee (A \vee B) \rightarrow C \vee (B \vee A)$ [repeated from page 9]

THEOREM 2: If $\vdash A \vee B$ then $\vdash B \vee A$

DEMONSTRATION:	1.	$A \vee B$	[Given]
	2.	$A \vee B \rightarrow B \vee A$	[A3]
	3.	$B \vee A$	[MP 1,2]

THEOREM 3: $\vdash \sim A \vee A$ or equivalently $\vdash A \rightarrow A$

DEMONSTRATION:	1.	$A \vee A \rightarrow A$	[A1]
	2.	$A \vee A \rightarrow A \dot{\rightarrow} \sim A \vee (A \vee A) \rightarrow \sim A \vee A$	[A4 A/A∨A B/A C/~A]
	3.	$\sim A \vee (A \vee A) \rightarrow \sim A \vee A$	[MP 1,2]
	4.	$A \rightarrow (A \vee A) \dot{\rightarrow} \sim A \vee A$	[3 def. of →]
	5.	$A \rightarrow A \vee A$	[A2]
	6.	$\sim A \vee A$	[MP 5,4]

THEOREM 4: If $\vdash A \rightarrow B$ and $\vdash B \rightarrow C$ then $\vdash A \rightarrow C$

DEMONSTRATION:	1.	$A \rightarrow B$	[Given]
	2.	$B \rightarrow C$	[Given]
	3.	$B \rightarrow C \dot{\rightarrow} \sim A \vee B \rightarrow \sim A \vee C$	[A4 A/B B/C C/~A]
	4.	$\sim A \vee B \rightarrow \sim A \vee C$	[MP 2,3]
	5.	$A \rightarrow B \dot{\rightarrow} A \rightarrow C$	[4 def. of →]
	6.	$A \rightarrow C$	[MP 1,5]

THEOREM 5A: $\vdash A \rightarrow \sim\sim A$

DEMONSTRATION:	1.	$\sim\sim A \vee \sim A$	[T3 A/~A]
	2.	$\sim A \vee \sim\sim A$	[T2 1]
	3.	$A \rightarrow \sim\sim A$	[2 def. of →]

THEOREM 5B: $\vdash \sim\sim A \to A$

DEMONSTRATION: 1. $\sim A \to \sim\sim\sim A$ [T5A A/~A]

2. $(\sim A \to \sim\sim\sim A) \to (A \vee \sim A \to A \vee \sim\sim\sim A)$ [A4 A/~A B/~ ~ ~A C/A]

3. $A \vee \sim A \to A \vee \sim\sim\sim A$ [MP 1, 2]

4. $A \vee \sim A$ [T3 and T2]

5. $A \vee \sim\sim\sim A$ [MP 4, 3]

6. $\sim\sim\sim A \vee A$ [T2 5]

7. $\sim\sim A \to A$ [6 def. of →]

DEFINITION: Wff A and B are <u>logically equivalent</u> if and only if $\vdash A \to B$ and $\vdash B \to A$.

When $\vdash A \to B$ and $\vdash B \to A$ we can write $A \equiv B$. Thus from theorems 5A and 5B we see that $A \equiv \sim\sim A$.

THEOREM 6: If $\vdash A \to B$ then $\vdash \sim B \to \sim A$

DEMONSTRATION: 1. $A \to \sim\sim A$ [T5A]

2. $B \to \sim\sim B$ [T5A A/B]

3. $A \to B$ [Given]

4. $A \to \sim\sim B$ [T4 3,2]

5. $\sim A \vee \sim\sim B$ [4 def. of →]

6. $\sim\sim B \vee \sim A$ [T2 5]

7. $\sim B \to \sim A$ [6 def. of →]

THEOREM 7A: $\vdash A \vee B \to \sim\sim A \vee \sim\sim B$

DEMONSTRATION: 1. $A \to \sim\sim A$ [T5A]

2. $(A \to \sim\sim A) \to (B \vee A \to B \vee \sim\sim A)$ [A4 B/~ ~A C/B]

3. $B \vee A \to B \vee \sim\sim A$ [MP 1,2]

4. $A \vee B \rightarrow B \vee A$ [A3]

5. $A \vee B \rightarrow B \vee \sim\sim A$ [T4 4,3]

6. $B \vee \sim\sim A \rightarrow \sim\sim A \vee B$ [A3 A/B B/ ~ ~ A]

7. $A \vee B \rightarrow \sim\sim A \vee B$ [T4 6,7]

8. $B \rightarrow \sim\sim B$ [T5A A/B]

9. $B \rightarrow \sim\sim B \;\dot{\rightarrow}\; \sim\sim A \vee B \rightarrow \sim\sim A \vee \sim\sim B$ [A4]

10. $\sim\sim A \vee B \rightarrow \sim\sim A \vee \sim\sim B$ [MP 8,9]

11. $A \vee B \rightarrow \sim\sim A \vee \sim\sim B$ [T4 7,10]

THEOREM 7B: $\vdash \sim\sim A \vee \sim\sim B \rightarrow A \vee B$

DEMONSTRATION: The demonstration of this theorem is similar to that of 7A.

THEOREM 8A: $\vdash A \vee (C \vee B) \rightarrow (A \vee C) \vee B$

DEMONSTRATION:
1. $A \vee (C \vee B) \rightarrow A \vee (B \vee C)$ [T1]

2. $C \rightarrow C \vee A$ [A2]

3. $C \vee A \rightarrow A \vee C$ [A3]

4. $C \rightarrow A \vee C$ [T4 2,3]

5. $(C \rightarrow A \vee C) \rightarrow (B \vee C \rightarrow B \vee (A \vee C))$ [A4]

6. $B \vee C \rightarrow B \vee (A \vee C)$ [MP 4,5]

7. $(B \vee C \rightarrow B \vee (A \vee C)) \rightarrow (A \vee (B \vee C) \rightarrow A \vee (B \vee (A \vee C)))$ [A4]

8. $A \vee (B \vee C) \rightarrow A \vee (B \vee (A \vee C))$ [MP 6,7]

9. $A \vee (B \vee (A \vee C)) \rightarrow (B \vee (A \vee C)) \vee A$ [A3]

10. $A \vee (B \vee C) \rightarrow (B \vee (A \vee C)) \vee A$ [T4 8,9]

11. $A \vee C \rightarrow (A \vee C) \vee B$ [A2]

12. $(A \vee C) \vee B \rightarrow B \vee (A \vee C)$ [A3]

13. $A \vee C \rightarrow B \vee (A \vee C)$ [T4 11,12]

14. $A \rightarrow A \vee C$ [A2]

15. $A \rightarrow B \vee (A \vee C)$ [T4 14,13]

16. $A \rightarrow B \vee (A \vee C) \dot{\rightarrow} ((B \vee (A \vee C)) \vee A \rightarrow (B \vee (A \vee C)) \vee (B \vee (A \vee C)))$
[A4 $B/B \vee (A \vee C)$ $C/B \vee (A \vee C)$]

17. $((B \vee (A \vee C)) \vee A) \rightarrow (B \vee (A \vee C) \dot{\vee} B \vee (A \vee C))$
[MP 15,16]

18. $(B \vee (A \vee C)) \vee (B \vee (A \vee C)) \rightarrow B \vee (A \vee C)$ [A1]

19. $(B \vee (A \vee C)) \vee A \rightarrow (B \vee (A \vee C))$ [T4 17,18]

20. $A \vee (B \vee C) \rightarrow B \vee (A \vee C)$ [T4 10,19]

21. $A \vee (C \vee B) \rightarrow B \vee (A \vee C)$ [T4 1,20]

22. $B \vee (A \vee C) \rightarrow (A \vee C) \vee B$ [A3]

23. $A \vee (C \vee B) \rightarrow (A \vee C) \vee B$ [T4 21,22]

THEOREM 8B: $\vdash (A \vee B) \vee C \rightarrow A \vee (B \vee C)$

DEMONSTRATION:

1. $C \vee (A \vee B) \rightarrow C \vee (B \vee A)$ [T1]

2. $(A \vee B) \vee C \rightarrow C \vee (A \vee B)$ [A3]

3. $(A \vee B) \vee C \rightarrow C \vee (B \vee A)$ [T4 2,1]

4. $C \vee (B \vee A) \rightarrow (C \vee B) \vee A$ [T8A]

5. $(A \vee B) \vee C \rightarrow (C \vee B) \vee A$ [T4 3,4]

6. $(C \vee B) \vee A \rightarrow A \vee (C \vee B)$ [A3]

7. $(A \vee B) \vee C \rightarrow A \vee (C \vee B)$ [T4 5,6]

8. $A \vee (C \vee B) \rightarrow A \vee (B \vee C)$ [T1]

9. $(A \vee B) \vee C \rightarrow A \vee (B \vee C)$ [T4 7,8]

THEOREM 8C: Let N be any natural number, and $B_1, B_2, \ldots, B_N$ any wff. Any two wff obtained by inserting parentheses in the expression $B_1 \vee B_2 \vee B_3 \vee \ldots \vee B_4$ are logically equivalent.

DEMONSTRATION: Mathematical induction is used for the demonstration. Theorems 8A and 8B show that the theorem holds for $N = 3$. Assume it holds for $N = K$ (induction assumption). Show it holds for $N = K + 1$. It is equivalent to show that any wff obtained by inserting parentheses in $B_1 \vee B_2 \vee \ldots \vee B_{K+1}$ is logically equivalent to $(B_1 \vee \ldots \vee B_K) \vee B_{K+1}$. Each wff has K connectives, one of which is the main connective. Let the $R^{th} \vee$ be the main connective. As the wff on both sides of the R^{th} connective have less than K connectives the parentheses may be left out (induction assumption). If $R = K$ the wff is already in the proper form. If not $(B_1 \vee \ldots \vee B_R) \vee (B_{R+1} \vee \ldots \vee B_{K+1})$

$$\equiv (B_1 \vee \ldots \vee B_R) \vee (B_{R+1} \vee \ldots \vee B_K) \vee B_{K+1}$$

$$\equiv (B_1 \vee \ldots \vee B_K) \vee B_{K+1}$$

Thus the theorem must be true for all N.

THEOREM 9A: $\vdash \sim(A \vee B) \rightarrow \sim A \wedge \sim B$

DEMONSTRATION: 1. $\sim \sim A \vee \sim\sim B \rightarrow A \vee B$ [T7B]

2. $\sim(A \vee B) \rightarrow \sim(\sim \sim A \vee \sim \sim B)$ [T6]

3. $\sim(A \vee B) \rightarrow (\sim A \wedge \sim B)$ [2 def. of $\wedge$]

THEOREM 9B: $\vdash \sim A \wedge \sim B \rightarrow \sim (A \vee B)$

DEMONSTRATION: The demonstration of this theorem is similar to that of 9A.

THEOREM 10: If $\vdash A \rightarrow B$ then $\vdash A \vee C \rightarrow B \vee C$

DEMONSTRATION: 1. $A \rightarrow B \overset{\bullet}{\rightarrow} C \vee A \rightarrow C \vee B$ [A4]

2. $A \rightarrow B$ [Given]

3. $C \vee A \rightarrow C \vee B$ [MP 2,1]

4. $A \vee C \rightarrow C \vee A$ [A3]

5. $A \vee C \rightarrow C \vee B$ [T4 4,3]

6. $C \vee B \rightarrow B \vee C$ [A3]

7. $A \vee C \rightarrow B \vee C$ [T4 5,6]

THEOREM 11A: $\vdash A \wedge (B \wedge C) \rightarrow (A \wedge B) \wedge C$

DEMONSTRATION: 1. $\sim\sim(\sim A \vee \sim B) \rightarrow (\sim A \vee \sim B)$ [T5B]

2. $\sim\sim(\sim A \vee \sim B) \vee \sim C \rightarrow (\sim A \vee \sim B) \vee \sim C$ [T10]

3. $(\sim A \vee \sim B) \vee \sim C \rightarrow \sim A \vee (\sim B \vee \sim C)$ [T8B]

4. $\sim\sim(\sim A \vee \sim B) \vee \sim C \rightarrow \sim A \vee (\sim B \vee \sim C)$ [T4 2,3]

5. $\sim B \vee \sim C \rightarrow \sim\sim(\sim B \vee \sim C)$ [T5A]

6. $\sim B \vee \sim C \rightarrow \sim\sim(\sim B \vee \sim C) \stackrel{\bullet}{\rightarrow} \sim A \vee (\sim B \vee \sim C) \rightarrow \sim A \vee \sim\sim(\sim B \vee \sim C)$ [A4]

7. $\sim A \vee (\sim B \vee \sim C) \rightarrow \sim A \vee \sim\sim(\sim B \vee \sim C)$ [MP 5,6]

8. $\sim\sim(\sim A \vee \sim B) \vee \sim C \rightarrow \sim A \vee \sim\sim(\sim B \vee \sim C)$ [T4 4,7]

9. $\sim(\sim A \vee \sim\sim(\sim B \vee \sim C)) \rightarrow \sim(\sim\sim(\sim A \vee \sim B) \vee \sim C)$ [T6 8]

10. $A \wedge (B \wedge C) \rightarrow (A \wedge B) \wedge C$ [9 def. of $\wedge$]

THEOREM 11B: $\vdash (A \wedge B) \wedge C \rightarrow A \wedge (B \wedge C)$

DEMONSTRATION: The demonstration of this theorem is similar to that of 8B.

THEOREM 11C: Let N be any natural number and $B_1, \ldots, B_N$ be any wff. Any two wff obtained by inserting parentheses in $B_1 \wedge B_2 \wedge \ldots \wedge B_N$ are logically equivalent.

DEMONSTRATION: The demonstration of this theorem is similar to that of 8C.

THEOREM 12A: $\vdash A \wedge B \rightarrow A$

DEMONSTRATION:

1. $A \rightarrow (A \vee \sim B)$ [A2]
2. $\sim A \vee (A \vee \sim B)$ [1 def. of $\rightarrow$]
3. $(A \vee \sim B) \vee \sim A$ [T2 2]
4. $(A \vee \sim B) \vee \sim A \rightarrow A \vee (\sim B \vee \sim A)$ [T8B]
5. $A \vee (\sim B \vee \sim A)$ [MP 3,4]
6. $A \vee (\sim B \vee \sim A) \rightarrow A \vee (\sim A \vee \sim B)$ [T1]
7. $A \vee (\sim A \vee \sim B)$ [MP 5,6]
8. $(\sim A \vee \sim B) \rightarrow \sim \sim (\sim A \vee \sim B)$ [T5]
9. $(\sim A \vee \sim B) \rightarrow \sim \sim (\sim A \vee \sim B) \;\dot{\rightarrow}\; A \vee (\sim A \vee \sim B) \rightarrow A \vee \sim \sim (\sim A \vee \sim B)$ [A4]
10. $A \vee (\sim A \vee \sim B) \rightarrow A \vee (\sim \sim (\sim A \vee \sim B))$ [MP 8,9]
11. $A \vee \sim \sim (\sim A \vee \sim B)$ [T4 7,10]
12. $\sim \sim (\sim A \vee \sim B) \vee A$ [T2 11]
13. $A \wedge B \rightarrow A$ [12 def. of $\wedge$ and $\rightarrow$]

THEOREM 12B: $\vdash A \wedge B \rightarrow B$

DEMONSTRATION:

1. $\sim B \vee \sim A \rightarrow \sim A \vee \sim B$ [A3]
2. $\sim (\sim A \vee \sim B) \rightarrow \sim (\sim B \vee \sim A)$ [T6 1]
3. $A \wedge B \rightarrow B \wedge A$ [2 def. of $\wedge$]
4. $B \wedge A \rightarrow B$ [T12A]
5. $A \wedge B \rightarrow B$ [T4 3,4]

THEOREM 13A: If $\vdash A \wedge B$ then $\vdash A$ and $\vdash B$

DEMONSTRATION:

1. $A \wedge B$ [Given]
2. $A \wedge B \rightarrow A$ [T12A]
3. A [MP 1,2]
4. $A \wedge B \rightarrow B$ [T12B]
5. B [MP 3,4]

THEOREM 13B: If $\vdash B_1 \wedge B_2 \wedge \ldots \wedge B_N$ then $\vdash B_1, \vdash B_2, \ldots, \vdash B_N$

DEMONSTRATION: Mathematical induction demonstrates this theorem. If $N = 2$ the theorem holds by theorem 13A. Assume the theorem holds for $N = K$. If $N = K + 1$ given that $\vdash B_1 \wedge \ldots \wedge B_K \wedge B_{K+1}$ we must show that $\vdash B_1, \ldots, \vdash B_{K+1}$. If $\vdash B_1 \wedge \ldots \wedge B_{K+1}$ then $\vdash B_1 \wedge (B_2 \wedge \ldots \wedge B_{K+1})$. By theorem 13A $\vdash B_1$ and $\vdash B_2 \wedge \ldots \wedge B_{K+1}$. From $\vdash B_2 \wedge \ldots \wedge B_{K+1}$ and the induction assumption $\vdash B_2, \vdash B_3, \ldots, \vdash B_{K+1}$. Hence the theorem holds for all N.

THEOREM 14A: If $\vdash A$ and $\vdash B$ then $\vdash A \wedge B$

DEMONSTRATION:

1.	$\sim(\sim A \vee \sim B) \vee (\sim A \vee \sim B)$	[T3]
2.	$(\sim A \vee \sim B) \vee \sim(\sim A \vee \sim B)$	[T2 1]
3.	$(\sim A \vee \sim B) \vee (A \wedge B)$	[2 def. of $\wedge$]
4.	$(\sim A \vee \sim B) \vee (A \wedge B) \rightarrow \sim A \vee (\sim B \vee (A \wedge B))$	[T8B]
5.	$\sim A \vee (\sim B \vee (A \wedge B))$	[MP 3,4]
6.	$A \rightarrow (B \rightarrow (A \wedge B))$	[5 def. of $\rightarrow$]
7.	A	[Given]
8.	$B \rightarrow A \wedge B$	[MP 7,6]
9.	B	[Given]
10.	$A \wedge B$	[MP 9,8]

THEOREM 14B: If $\vdash B_1, \vdash B_2, \ldots, \vdash B_N$ then $\vdash B_1 \wedge \ldots \wedge B_N$

DEMONSTRATION: This theorem also can be shown by mathematical induction.

THEOREM 15: If $\vdash A \rightarrow B$ and $\vdash C \rightarrow D$ then $\vdash A \wedge C \rightarrow B \wedge D$

DEMONSTRATION:

1.	$A \rightarrow B$	[Given]
2.	$C \rightarrow D$	[Given]
3.	$\sim B \rightarrow \sim A$	[T6 1]

4. $\sim D \rightarrow \sim C$ [T6 2]

5. $\sim D \rightarrow \sim C \,\dot{\rightarrow}\, \sim B \vee \sim D \rightarrow \sim B \vee \sim C$ [A4]

6. $\sim B \vee \sim D \rightarrow \sim B \vee \sim C$ [MP 4,5]

7. $\sim B \vee \sim C \rightarrow \sim A \vee \sim C$ [T10 3]

8. $\sim B \vee \sim D \rightarrow \sim A \vee \sim C$ [T4 6,7]

9. $\sim(\sim A \vee \sim C) \rightarrow \sim(\sim B \vee \sim D)$ [T6 8]

10. $A \wedge C \rightarrow B \wedge D$ [9 def. of $\wedge$]

THEOREM 16A: $\vdash A \vee (B \vee C) \rightarrow (A \vee B) \wedge (A \vee C)$

1. $B \wedge C \rightarrow B,\ B \wedge C \rightarrow C$ [T12A, T12B]

2. $B \wedge C \rightarrow B \,\dot{\rightarrow}\, A \vee (B \wedge C) \rightarrow A \vee B,$
   $B \wedge C \rightarrow C \,\dot{\rightarrow}\, A \vee (B \wedge C) \rightarrow A \vee C$ [A4]

3. $A \vee (B \wedge C) \rightarrow A \vee B,\ A \vee (B \wedge C) \rightarrow A \vee C$ [MP 1,2]

4. $(A \vee (B \wedge C)) \wedge (A \vee (B \wedge C)) \rightarrow (A \vee B) \wedge (A \vee C)$ [T15 3]

5. $\sim\sim(A \vee (B \wedge C)) \rightarrow \sim(\sim(A \vee (B \wedge C)) \vee \sim(A \vee (B \wedge C)))$
   [A1 $A/A \vee (B \wedge C)$, T6]

6. $A \vee (B \wedge C) \rightarrow \sim\sim(A \vee (B \wedge C))$ [T5A]

7. $(A \vee (B \wedge C)) \rightarrow (A \vee (B \wedge C)) \wedge (A \vee (B \wedge C))$ [T4 6,5 def. of $\wedge$]

8. $(A \vee (B \wedge C)) \rightarrow (A \vee B) \wedge (A \vee C)$ [T4 7,4]

THEOREM 16B: $\vdash (B \wedge C) \vee A \rightarrow (B \vee A) \wedge (C \vee A)$

DEMONSTRATION: The demonstration follows that of 16A except that in 3-6, where A4 and MP are used, only T10 is used.

THEOREM 17: $A \,\dot{\rightarrow}\, B \rightarrow C \equiv A \wedge B \rightarrow C$

DEMONSTRATION: 1. $A \,\dot{\rightarrow}\, B \rightarrow C = \ \sim A \,\dot{\vee}\, \sim B \vee C$ [def. of $\rightarrow$]

$\equiv \ \sim A \vee \sim B \,\dot{\vee}\, C$ [T8]

2. $A \vee \sim B \equiv \sim\sim(\sim A \vee \sim B)$ [T5]

3. $\sim A \vee \sim B \,\dot{\vee}\, C \equiv \sim\sim(\sim A \vee \sim B) \vee C$ [T10 2]

4. $A \dot{\rightarrow} B \rightarrow C \equiv \sim\sim(\sim A \vee \sim B) \vee C$ [T4 1,3]

5. $A \dot{\rightarrow} B \rightarrow C \equiv (A \wedge B) \rightarrow C$ [def. of $\wedge$, $\rightarrow$]

THEOREM 18: $A \wedge B \equiv B \wedge A$

DEMONSTRATION:

1. $\sim A \vee \sim B \equiv \sim B \vee \sim A$ [A3]

2. $\sim(\sim A \vee \sim B) \equiv \sim(\sim B \vee \sim A)$ [T6]

3. $A \wedge B \equiv B \wedge A$ [2 def. of $\wedge$]

THEOREM 19: $\vdash B \wedge (A \vee C) \rightarrow A \vee (B \wedge C)$

DEMONSTRATION:

1. $(\sim B \vee \sim C) \vee \sim(\sim B \vee \sim C)$ [T3 T2]

2. $\sim B \vee (\sim C \vee \sim(\sim B \vee \sim C))$ [T8 MP]

3. $B \rightarrow (C \rightarrow (B \wedge C))$ [def. of $\rightarrow$ and $\wedge$]

4. $C \rightarrow B \wedge C \dot{\rightarrow} A \vee C \rightarrow A \vee (B \wedge C)$ [A4]

5. $B \rightarrow (A \vee C \rightarrow A \vee (B \wedge C))$ [T4 3,4]

6. $B \wedge (A \vee C) \rightarrow A \vee (B \wedge C)$ [T17]

THEOREM 20: $\vdash (A \vee B) \wedge (A \vee C) \rightarrow A \vee (B \wedge C)$

DEMONSTRATION:

1. $(A \vee B) \wedge (A \vee C) \rightarrow A \vee ((A \vee B) \wedge C)$ [T19]

2. $C \wedge (A \vee B) \rightarrow A \vee (C \wedge B)$ [T19]

3. $C \wedge (A \vee B) \rightarrow A \vee (C \vee B) \dot{\rightarrow} A \vee (C \wedge (A \vee B)) \rightarrow A \vee (A \vee (C \wedge B))$ [A4]

4. $A \vee (C \wedge (A \vee B)) \rightarrow A \vee A \vee (C \wedge B)$ [MP 2,3]

5. $A \vee A \rightarrow A$ [A1]

6. $A \vee A \vee (C \wedge B) \rightarrow A \vee (C \wedge B)$ [T10 5]

7. $A \vee (C \wedge (A \vee B) \rightarrow A \vee (C \wedge B)$ [T4 4,6]

8. $C \wedge (A \vee B) \equiv (A \vee B) \wedge C$ [T18]

9. $A \vee (C \wedge (A \vee B)) \equiv A \vee ((A \vee B) \wedge C)$ [A4, MP]

10. $A \vee ((A \vee B) \wedge C) \rightarrow A \vee (C \wedge B)$ [MP 9,7]

11. $(A \vee B) \wedge (A \vee C) \rightarrow A \vee (B \wedge C)$ [T4, 1,10]

THEOREM 21A: $A\vee(B_1\wedge B_2\wedge\ldots\wedge B_N) \equiv (A\vee B_1)\wedge(A\vee B_2)\wedge\ldots\wedge(A\vee B_N)$

THEOREM 21B: $(B_1\wedge\ldots\wedge B_N)\vee A \equiv (B_1\vee A)\wedge(B_2\vee A)\wedge\ldots\wedge(B_N\vee A)$

DEMONSTRATION: These two theorems can be proved by mathematical induction.

I. PRIME WFF

This section and the next introduce and examine two concepts that facilitate the demonstration of the completeness of propositional calculus. These two concepts are prime wff and wff in conjunctive normal form.

DEFINITION: A prime wff is a wff in which the only connectives are $\sim$ and $\vee$, and $\sim$ is prefixed only to atomic wff. Therefore a wff A is prime when it is in the form $B_1\vee B_2\vee\ldots\vee B_N$ where the B_i are atomic wff or equal $\sim C_i$ where C_i is an atomic wff. The B_i are called the disjuncts of A.

A prime wff A is provable if there is some atomic wff C such that both C and $\sim C$ are disjuncts of A. That the two disjuncts C and $\sim C$ determine its provability is seen in theorem 3, $\vdash A\vee\sim A$. That additional disjuncts do not alter its provability is seen by repeated applications of the second axiom and Modus Ponens:

1. $\sim C\vee C$ [T3]
2. $(\sim C\vee C)\rightarrow(\sim C\vee C)\vee B_3$ [A2]
3. $\sim C\vee C\vee B_3$ [MP 1,2]
4. $\sim C\vee C\vee B_3\rightarrow(\sim C\vee C\vee B_3)\vee B_4$ [A2]
5. $\sim C\vee C\vee B_3\vee B_4$ [MP 3,4]

If there is no atomic wff C such that both C and $\sim C$ are disjuncts of a prime wff A, A is not provable. A, equal to $B_1 \vee B_2 \vee \ldots \vee B_N$, can be seen to be not true by the following assignment of the atomic wff C_i to $\{T, F\}$:

$$H(C_i) = \begin{cases} T \text{ if } B_i = \sim C_i \\ F \text{ if } B_i = C_i \end{cases}$$

In this way each disjunct, and hence the entire prime wff, will be mapped onto F. Since all provable wff were shown to be true, A, which is not true, could not be provable. Therefore a prime wff is provable if and only if there is an atomic wff C such that C and $\sim C$ are disjuncts.

J. WFF IN CONJUNCTIVE NORMAL FORM

The purpose of this section is to define conjunctive normal form and to show that any wff can be written in this form.

DEFINITION: A <u>wff in conjunctive normal form (CNF)</u> is a wff in the form $B_1 \wedge B_2 \wedge \ldots \wedge B_N$ where each B_i is a prime wff.

If not all wff have a logically equivalent wff in CNF, then there must be a smallest wff (wff with the fewest connectives) that could not be put in the form. Let A be such a wff. Assume that A is in a form using only the formal symbols. Since an atomic wff is in CNF, A must have a main connective. Since only formal symbols are used the main connective must be either $\vee$ or $\sim$.

1. The main connective is $\vee$.

Then there are wff B and C such that A is $B \vee C$. Since B

and C have one less connective they can be written in CNF or A is $(B_1 \wedge \ldots \wedge B_N) \vee (C_1 \wedge \ldots \wedge C_M)$ where the B_i and C_i are prime wff. By theorem 21 this is logically equivalent to $((B_1 \wedge \ldots \wedge B_N) \vee C_1) \wedge ((B_1 \wedge \ldots \wedge B_N) \vee C_2) \wedge \ldots \wedge ((B_1 \wedge \ldots \wedge B_N) \vee C_M)$. Also by theorem 21 $(B_1 \wedge \ldots \wedge B_N) \vee C_K$ is logically equivalent to $(B_1 \vee C_K) \wedge (B_2 \vee C_K) \wedge \ldots \wedge (B_N \vee C_K)$. Thus A is logically equivalent to a wff in CNF.

2. The main connective is $\sim$.

Then there is a wff B such that A is $\sim B$. Since if B were atomic $\sim B$ would be in CNF, B must be composite.

2a. The main connective of B is $\sim$.

Then $A = \sim\sim C$ when $B = \sim C$. But then $A \equiv C$ since by theorem 6 $C \equiv \sim\sim C$. Since C has two fewer connectives than A, C can be put in CNF.

2b. The main connective of B is $\vee$.

Then $A = \sim(C \vee D)$ where $B = C \vee D$. From theorems 9A and 9B we see that $\sim(C \vee D) \equiv \sim C \wedge \sim D$. $\sim C$ and $\sim D$ must have at least one fewer connective than A. Thus $\sim C$ and $\sim D$ can be put in CNF. Therefore $A \equiv \sim C \wedge \sim D$ can be written in CNF.

Thus any wff can be written in CNF.

K. TRUE WFF ARE PROVABLE

Let A be a true wff. We can now show that this wff is also provable. Assume for contradiction that A is not provable. A can be written in CNF. Let $A \equiv C_1 \wedge C_2 \wedge \ldots \wedge C_N$ where the C_i are prime wff. This means $\vdash A \rightarrow C_1 \wedge \ldots \wedge C_N$ and $\vdash C_1 \wedge \ldots \wedge C_N \rightarrow A$. If A is not provable then $C_1 \wedge \ldots \wedge C_N$ is not provable. From theorem 14 we know if

$\vdash C_1, \ldots, \vdash C_N$ then $\vdash C_1 \wedge \ldots \wedge C_N$. So if $C_1 \wedge \ldots \wedge C_N$ is not provable one of the C_i's must not be provable. Then, from the discussion on prime wff, there is no atomic wff X such that both X and ~X are in C_i. (If there were such an atomic wff C_i would be provable.) But if there is no such atomic wff, C_i was shown to be not true. And if any of the C_i's are not true then $C_1 \wedge C_2 \wedge \ldots \wedge C_N$ is not true. But we are given that $C_1 \wedge \ldots \wedge C_N$ is true. Therefore our assumption that A is not provable must be false. Hence, A is provable if and only if A is true.

The power of this statement is evident. It is now possible to check a wff, A, in a purely mechanical fashion to see if it is provable. One simply needs to assign all the possible combinations of T and F to the atomic wff in A. If G(A) (G was defined in section on true wff) is T in all cases, the wff A is true and provable.

CHAPTER IV

PREDICATE CALCULUS

The predicate calculus adds the notions of propositional functions and quantifiers to the concepts in propositional calculus. A propositional function is a proposition which contains variables. A propositional function cannot be considered valid or invalid until a system of values are assigned to the variables. The universal quantifier, $\forall$, is a connective in the predicate calculus.

The ultimate goal of this chapter will be to show that the predicate calculus is complete. This will be done by showing that wff in the predicate calculus are true if and only if they are provable.

A. BASICS

Four types of formal symbols are needed for the predicate calculus. First we need an infinite stock of objects called individuals -- x, y, z, etc. Then we require an infinite set of predicates — F, G, H, etc. A natural number is assigned to each predicate and called the <u>order</u> of the predicate. The connectives of this formal language are $\sim$, $\vee$ and $\forall$. As in the proportional calculus the formal symbols (and) are needed for punctuation.

As before a finite string of the formal symbols is called an <u>expression</u>. An individual x is considered <u>free</u> in the expression if x occurs in the expression and $\forall x$ does not. An individual x is

<u>bound</u> in an expression if ∀x occurs in the expression. For example y is free in yF~ and Fy∀ ~Gxy but bound in ∀y and ~(∀yFxy).

In predicate calculus we are still concerned with well formed formula (wff). The definition of an atomic wff is modified to accommodate the notion of propositional functions. The definition of wff is modified to include the quantifier.

DEFINITION: An <u>atomic wff</u> is an expression of the form $(Gx_1\ x_2 \ldots x_N)$ where G is a predicate of order N and the x_i are any N individuals.

DEFINITION: A <u>wff</u> is an expression with one of the following four forms:

1. A where A is an atomic wff
2. (~A) where A is a wff
3. (A∨B) where A and B are wff in which there is no individual bound in one and free in the other
4. (∀tD) where D is a wff and t is any individual free in D.

In the wff (∀tD), D is the <u>scope</u> of the ∀t. Each wff which is not atomic is composite and possess a main connective. If the main connective of a wff A is ~ then there is a wff B such that A is (~B); if the main connective is ∨, there are wff B and C such that A is (B∨C); if it is ∀, there is a wff D and an individual t such that A is (∀tD). [3]

B. PARENTHESES OMITTING CONVENTIONS

The same conventions in propositional calculus are utilized here. The symbols ∧, → and ↔ are defined as before. A new symbol , ∃ , can also be used. The wff $(\sim(\forall t(\sim A)))$ can be written in the abbreviated form $(\exists tA)$. The symbols $\sim, \vee, \wedge, \rightarrow$ and $\leftrightarrow$ have the same relative bracketing power which is strengthened as before, by adding a dot or several dots over the connective. The connectives $\forall$ and $\exists$ have the weakest bracketing power of all the connectives. Let $Q_1, Q_2 \ldots Q_N A$ be a sequence such that Q_i is either $\forall t_i$ or $\exists t_i$ and A is a wff in which $t_1, \ldots, t_N$ are free individuals. Then $Q_1\, Q_2 \ldots Q_N A$ denotes that wff $(Q_1\, (Q_2\, (Q_3 \ldots (Q_N A))))$ whose main connective is in Q_1.

C. SYNTACTICAL TRANSFORMS

The following two syntactical transforms, or mappings of the set of all wff into the set of all wff, will be utilized in the definitions of true and provable wff.

The first syntactical transform I_t^s switches the individuals s and t throughout the wff. More precisely:

1. $I_t^s\,(H\, x_1 \ldots x_N) = H_{z_1} \ldots z_N$ where $z_i = \begin{cases} s & \text{if } x_i = t \\ t & \text{if } x_i = s \\ x_i & \text{otherwise} \end{cases}$

2. $I_t^s\,(\sim A) = \sim I_t^s\,(A)$

3. $I_t^s\,(A \vee B) = (I_t^s A) \vee (I_t^s B)$

4. $I_t^s\,(\forall uA) = \begin{cases} \forall s I_t^s A & \text{if } u = t \\ \forall t I_t^s A & \text{if } u = s \\ \forall u I_t^s A & \text{otherwise}. \end{cases}$

The effects of the second syntactical transform S_t^s are not quite as readily apparent.

S_t^s is defined as follows:

1. $S_t^s\ (Hx_1 \ldots x_N) = H_{z_1} \ldots z_N$ where $z_i = \begin{cases} s & \text{if } x_i = t \\ x_i & \text{otherwise} \end{cases}$

2. $S_t^s\ (\sim A) = \sim (S_t^s A)$

3. $S_t^s\ (A \vee B) = (S_t^s A) \vee (S_t^s B)$

4. $S_t^s\ (\forall t A) = \forall t A$

5. $S_t^s\ (\forall s A) = \forall t I_t^s A$

6. $S_t^s\ (\forall u A) = \forall u S_t^s A,\ s \neq u \neq t.$

Each free instance of t and each bound instance of t in the scope of $\forall s$ are changed to s. Each bound s not in the scope of $\forall t$ is changed to t. [3]

D. TRUE WFF

The vehicle for defining truth is quite different in the two calculi presented here. In propositional calculus wff are mapped onto {T,F}. In predicate calculus wff are mapped into structures.

DEFINITION: A <u>structure</u> is an ordered N-tuple. The first term is a non-empty set called the <u>basic set</u>. The remaining terms are either relations of the basic set or displayed members of the basic set.

In the structure

(I) ({a,b,c},{(a), (c), (b)}, {(a,b), (a,a)}, {(a,b,c)}, a,b)

{a,b,c} is the basic set. In this example there are three relations, the unary relationship $\{(a),(b),(c)\}$, R_1, the binary relationship $\{(a,b), (a,a)\}$, R_2, and the trinary relationship $\{(a,b,c)\}$, R_3. The displayed members of the basic set are a and b.

DEFINITION: A wff A is defined in a structure S if and only if each predicate of order N in A can be associated with a N^{ary} relation in S and each free individual in A can be associated with a displayed member of the basic set of S.

This association can be denoted by means of a mapping. Consider the wff

(II) $(Fx \vee \sim Gxy) \rightarrow \forall z\ (Hz \wedge Fy)$

This wff is defined in (I). The predicates of order one can be mapped on the unary relationship - $f(F) = \{(a),(b),(c)\}, = R_1$, $f(H) = \{(a),(b),(c)\} = R_1$. The predicate of order two can be mapped on the binary relationship - $f(G) = \{(a,b)\ (a,a)\} = R_2$. Lastly the free individuals, x and y, can be mapped on the displayed members of the basic set, a and b. There are many different ways $\{x,y\}$ can be mapped into $\{a,b\}$; one is $f(x) = a$, $f(y) = a$. The wff Axyxz cannot be defined in (I) because there is no quadruple relation.

We need to define what propositions are, within the scope of the structure. We will call such propositions structural well-formed formulae or swff. Let S be any structure with a basic set B and a N^{ary} relationship R. $(R\ a_1\ a_2 \ldots a_N)$ is a swff when $(a_1, \ldots, a_N)$ is an N-tuple of members of B. If A and C are swff then $(\sim A)$ and $(A \vee C)$ are swff. For the last form in which swff appear we must extend the notion of S^s_t to apply to swff. When s and t are members of the basic set and A a swff, $S^s_t A$ replaces all instances of t with s. If A is a swff, and s and a are members of the basic set, a occurring in A and s not occurring in A, then $(\forall s\ S^{s}_{a}\ A)$ is a swff.

If a wff A is defined in a structure S under a mapping f, then the mapping can be extended to map all the connective and punctuation symbols onto themselves so that f maps the wff A onto a swff, f(A). Thus from the earlier example where the wff (II) was defined in the structure (I) under the mapping f,

(III) $$f((Fx\vee \sim Gxy) \to \forall z(Hz\wedge Fy))$$

$$= (R_1 a\vee \sim R_2 aa) \to \forall z(R_1 z\wedge R_1 a).$$

Next we are concerned with whether or not a certain relationship exists between a swff and a structure. The definition of a swff <u>holding</u> in a structure S with a N^{ary} relation R and members of the basic set $b,c,a_1,a_2,\ldots,a_N$ follows.

1. The swff $Ra_1\ldots a_N$ holds in S if and only if $(a_1,\ldots,a_N)\in R$.
2. The swff $\sim A$ holds in S if and only if A does not hold in S.
3. The swff $B\vee C$ holds in S if and only if B holds in S or C holds in S.
4. The swff $\forall sA$ holds in S if and only if $S^{a}_{s}A$ holds in S whenever a is a member of the basic set.

Thus the swff (III) holds in the structure (I). $[(R_1 a\vee \sim R_2 aa) \to \forall z(R_1 z\wedge R_1 a)]$ is $[\sim(R_1 a\vee \sim R_2 aa)\vee\forall z(R_1 z\wedge R_1 a)]$ and since $R_1 a$, $R_1 b$ and $R_1 c$ hold in S, $\forall z(R_1 z\wedge R_1 a)$ holds in S.

A structure S is a <u>model</u> of a wff A under a mapping f if and only if A is defined in S under f and the swff f(A) holds in S. A structure S is a model of a set K of wff under a mapping f if and only if S is a model for each member of K under f.

We are finally ready to define true wff. A wff A is <u>true</u> if and only if S is a model of A under f whenever A is defined in S under f. Thus $Fx \vee \sim Fx$ is true since $f(Fx \vee \sim Fx)$ is $f(Fx) \vee f(\sim Fx)$ and if $f(Fx)$ does not hold in a structure in which Fx is defined $f(\sim Fx)$ will. [3]

E. PROVABLE WFF

The definitions for provable wff in the two calculi are quite similar. In the predicate calculus to be developed here there is one more axiom and one more rule of inference than in the propositional calculus as developed in Chapter III.

A <u>proof</u> is a finite sequence of wff such that each term E possesses one of the five forms:

I
1. $A \vee A \rightarrow A$
2. $A \rightarrow A \vee B$
3. $A \vee B \rightarrow B \vee A$
4. $A \rightarrow B \dot{\rightarrow} C \vee A \rightarrow C \vee B$
5. $\forall t A \rightarrow S_t^s A$, t free in A, s not bounded in $\forall t A$

or II
1. E is preceded by wff of the form D and $D \rightarrow E$ (Modus Ponens).
2. E has the form $A \rightarrow \forall t B$ and is preceded by $A \rightarrow B$. [3]

The wff under I are axiom schemes for the predicate calculus and the statements under II are rules of inference.

A wff C is provable if and only if there is a proof whose last term is C. Once again we denote this by $\vdash C$.

The principle which follows helps to demonstrate the properties of provable wff.

THE INDUCTION PRINCIPLE FOR PROVABLE WFF: Each provable wff has property P provided that:

1. each member of the axiom scheme has property P
2. if A and $A \to B$ are both provable and have property P then so does B
3. if $A \to B$ is provable and t is any individual free in B but not occurring in A, then $A \to \forall tB$ has the property P.

DEMONSTRATION: Let P be a property satisfying the above assumptions. Assume for contradiction that there is a provable wff A which does not have the property P. Consider the proof of A; let B be the first wff in the proof of A that does not have property P. By 1, B cannot be an axiom. From the definition of proof we see that B must be justified by one of the rules of inference. If by Modus Ponens , then there are wff D and $D \to B$ that precede B. Since they precede B they are provable and have property P. Therefore by 2, B must have property P. If the second rule of inference is the justification for B then B has the form $D \to \forall tE$ and is preceded by $D \to E$. Since $D \to \forall tE$ is a wff, t must be free in E and not occurring in D. Since $D \to E$ precedes B it must be provable and have property P.

Therefore by 3,$D \to \forall tE$ must have property P.

This contradiction establishes the induction principle. [3]

F. PROVABLE WFF ARE TRUE

We will utilize the induction principle for provable wff to show that all provable wff are true. The three parts of the induction principle will be demonstrated in the three lemmas.

LEMMA 1: All axioms are true.

DEMONSTRATION: AXIOM 1: $A \vee A \to A$

We must show that whenever the wff $A \vee A \to A$ is defined in a structure S under a mapping f then the swff $f(A \vee A \to A)$ holds in S. Since $f(A \vee A \to A) = f(A \vee A) \to f(A)$ we need only show that either $f(A \vee A)$ does not hold or that $f(A)$ does hold. If $f(A)$ does not hold then $f(A) \vee f(A) = f(A \vee A)$ does not hold. If $f(A \vee A)$ holds then $f(A) \vee f(A)$ must hold so $f(A)$ must hold. Therefore $A \vee A \to A$ must be true.

AXIOM 2: $A \to A \vee B$

We must show that if $A \to A \vee B$ is defined in a structure S under a mapping f then the swff $f(A \to A \vee B)$ must hold. In order for $f(A \to A \vee B)$ to hold,if $f(A)$ holds then $f(A \vee B)$ must hold. $f(A \vee B)$ holds if $f(A)$ holds or $f(B)$ holds. Therefore $A \to A \vee B$ must be true.

AXIOM 3: $A \vee B \to B \vee A$

We must show when $A \vee B \to B \vee A$ is defined in a structure S under a mapping f then the swff $f(A \vee B \to B \vee A)$ holds in S. $f(A \vee B \to B \vee A) =$

$f(\sim(A \vee B) \vee (B \vee A))$ so we must show either $f(\sim(A \vee B))$ or $f(B \vee A)$ holds. If $f(B \vee A)$ doesn't hold then $f(A \vee B)$ doesn't hold and $f(\sim(A \vee B))$ holds. If $f(\sim(A \vee B))$ doesn't hold then $f(A \vee B)$ and $f(B \vee A)$ would hold. Therefore $f(A \vee B \rightarrow B \vee A)$ always holds.

AXIOM 4: $A \rightarrow B \dot{\rightarrow} C \vee A \rightarrow C \vee B$

We must show that $f(A \rightarrow B \dot{\rightarrow} C \vee A \rightarrow C \vee B)$ holds in a structure S when $A \rightarrow B \dot{\rightarrow} C \vee A \rightarrow C \vee B$ is defined in S under f. $f(A \rightarrow B \dot{\rightarrow} C \vee A \rightarrow C \vee B) = f(A \rightarrow B) \dot{\rightarrow} f(C \vee A \rightarrow C \vee B)$ holds only if $f(C \vee A \rightarrow C \vee B)$ holds whenever $f(A \rightarrow B)$ holds. Given that $f(A \rightarrow B)$ holds, $f(B)$ holds whenever $f(A)$ holds. But this means $f(C \vee B)$ holds whenever $f(C \vee A)$ holds. Therefore $f(C \vee A \rightarrow CNB)$ holds whenever $f(A \rightarrow B)$ holds and $A \rightarrow B \dot{\rightarrow} C \vee A \rightarrow C \vee B$ is a true wff.

AXIOM 5: $\forall tA \rightarrow S^s_t A$ (t free in A, s not bound in A)

We must show $f(\forall tA \rightarrow S^s_t A)$ holds in a structure S whenever $\forall tA \rightarrow S^s_t A$ is defined in S under f. But $f(\forall tA)$ holds if and only if $f(S^s_t A)$ holds. Therefore $f(S^s_t A)$ holds whenever $f(\forall tA)$ holds and $\forall tA \rightarrow S^s_t A$ is true.

Thus all five axioms are true.

LEMMA 2: If A and $A \rightarrow B$ are true then B is true.

DEMONSTRATION: We want to show that $f(B)$ holds in each structure S in which B is defined under f.

If A is defined in S then $f(A)$ and $f(A \rightarrow B)$ hold in S. Since $f(A \rightarrow B) = f(\sim A \vee B) = f(\sim A) \vee f(B)$ and $f(\sim A)$ does not hold, $f(B)$ must hold.

If A cannot be defined in S under f then there must be predicates or free individuals in A that are not assigned a value under f. Extend S and f to S^1 and f^1 in such a manner that for every predicate P_i without an image, a relation H_i is added to S and $f(P_i) = H_i$. If there are free individuals x_i in A without an image in S let a be a member of the basic set that is displayed in S^1 so that $f^1(x_i) = a$. Note that all the assignments for f still hold for f^1. Now A is defined in S^1 under f^1 so $A \to B$ must be defined in S^1 under f^1. And since A and $A \to B$ are true $f^1(A)$ and $f^1(A \to B)$ must hold in S^1. Therefore $f^1(B)$ holds in S^1. But $f^1(B) = f(B)$ and so $f(B)$ holds in S^1 and in S.

Thus if A and $A \to B$ are true then B is true.

LEMMA 3: $A \to \forall tB$ is true provided $A \to B$ is true, t is free in B and t does not occur in A.

DEMONSTRATION: We are given that $A \to B$ is true and $A \to \forall tB$ is a wff. We want to show $A \to \forall tB$ is true. Let S be any structure in which $A \to \forall tB$ is defined under f. Since $f(A \to \forall tB)$ is $f(A) \to f(\forall tB)$ if $f(A)$ holds then we must show $f(\forall tB)$ holds. Thus we need to show that $f(S_t^aB)$ holds for any a which is a member of the basic set of S, given that $A \to B$ is true, $A \to \forall tB$ is a wff and $f(A)$ holds in S. Let S^1 be the same as S but with the addition of a as displayed member of the basic set. Then let $f^1(x)$, where x is an individual in A or B, equal $f(x)$ if $x \neq t$ and equal a if $x=t$. Since $A \to B$ is true and defined in S^1 under f^1, $f^1(A \to B)$ holds in S^1. But $f^1(A \to B) = f^1(A) \to f^1(B)$ and $f^1(A) = f(A)$ since t does not occur in A. Thus $f^1(B)$ must hold. But $f^1(B) = f(S_t^aB)$ so $f(S_t^aB)$ must hold and $\forall tB$ must be true.

Thus from the induction principle for provable wff and the three preceding lemmas we see that all provable wff are true.

G. USEFUL THEOREMS AND CONCEPTS

The following theorems will be utilized in the demonstration of the completeness of the predicate calculus.

THEOREM 1: If $A_1, A_2, \ldots, A_N$ is a proof, then $I_t^s A_1, I_t^s A_2, \ldots, I_t^s A_N$ is a proof when s and t are individuals.

DEMONSTRATION: We are given that $A_1, \ldots, A_N$ is a proof. Assume for contradiction that $I_t^s A, \ldots, I_t^s A_N$ is not a proof. Then there is a first wff $I_t^s A_k$ that is not provable. There are three possibilities for A_k and hence $I_t^s A_k$.

1. A_k is a member of the axiom set.

We shall show for each axiom that if A_k is an axiom then $I_t^s A_k$ is an axiom.

$$I_t^s(A \vee A \to A) = I_t^s A \vee I_t^s A \to I_t^s A$$

$$I_t^s(A \to A \vee B) = I_t^s A \to I_t^s A \vee I_t^s B$$

$$I_t^s(A \vee B \to B \vee A) = I_t^s A \vee I_t^s B \to I_t^s B \vee I_t^s A$$

$$I_t^s(A \to B \mathbin{\dot{\to}} C \vee A \to C\ B) = I_t^s A \to I_t^s B \mathbin{\dot{\to}} I_t^s C \vee I_t^s A \to I_t^s C \vee I_t^s B$$

The fifth axiom must be considered in four separate situations.

If s, t, u and v are all different individuals $I_t^s(\forall u A \to S_u^v A) = \forall u I_t^s A \to I_t^s S_u^v A = \forall u I_t^s A \to S_u^v I_t^s A$.

If s, t, u are all different individuals $I^s_t(\forall uA \rightarrow S^s_u A) = \forall u I^s_t A \rightarrow I^s_t A \rightarrow I^s_t S^s_u A = \forall u I^s_t A \rightarrow S^t_u I^s_t A$ because neither s nor u are bounded in A.

If s, t, v are all different individuals $I^s_t(\forall tA \rightarrow S^v_t A) = \forall_s I^s_t A \rightarrow I^s_t S^v_t A = \forall_s I^s_t \rightarrow S^v_s I^s_t A$.

If only s and t are different individuals $I^s_t(\forall tA \rightarrow S^s_t A) = \forall_s I^s_t A \rightarrow I^s_t S^s_t A = \forall_s I^s_t A \rightarrow S^t_s I^s_t A$.

Thus if A_k is an axiom so is $I^s_t A_k$.

2. A_k is preceded by A_i and $A_i \rightarrow A_k$.

Then $I^s_t A_k$ is preceded by $I^s_t A_i$ and $I^s_t(A_i \rightarrow A_k) = I^s_t A_i \rightarrow I^s_t A_k$ and hence is provable by Modus Ponens.

3. $A_k = B \rightarrow \forall uC$ and is preceded by $B \rightarrow C$.

Then $I^s_t(B \rightarrow \forall uC) = I^s_t B \rightarrow I^s_t \forall uC$ is preceded by $I^s_t(B \rightarrow C) = I^s_t B \rightarrow I^s_t C$. If u does not equal s or t then $I^s_t A_k = I^s_t B \rightarrow \forall u I^s_t C$ and is justified by the second rule of inference. If u equals s then $I^s_t A_k = I^s_t B \rightarrow \forall t I^s_t C$ and is justified by the second rule of inference. Thus we have contradicted our assumption that $I^s_t A_k$ is not provable and established the theorem. [3]

The next three theorems are presented because they will be used to prove the completeness of predicate calculus. Their demonstration is not included because their proofs are so similar to the proofs in the chapter on propositional calculus.

THEOREM 2: If $\vdash A \rightarrow B$ and $\vdash C \rightarrow D$ then $\vdash A \wedge C \rightarrow B \wedge D$

THEOREM 3: $\vdash B \wedge \sim B \rightarrow A$

THEOREM 4: If $\vdash \sim A \rightarrow A$ then $\vdash A$

The rest of this section will be concerned with consequences of a set of wff. If K is a set of wff and B, a wff, then "B is a consequence of K" is denoted by "$K \vdash B$." B is said to be a <u>consequence of</u> or deducible from, K if there is a non-empty finite subset of K, $\{A_1,\ldots,A_N\}$, such that $\vdash A_1 \wedge \ldots \wedge A_N \rightarrow B$.

THEOREM 5: If $K \vdash A$ and $K \vdash B$ then $K \vdash A \wedge B$

DEMONSTRATION: Since $K \vdash A$ and $K \vdash B$ there are subsets of K, $\{A_1,\ldots,A_N\}$ and $\{B_1,\ldots,B_M\}$ such that $\vdash A_1 \wedge \ldots \wedge A_N \rightarrow A$ and $\vdash B_1 \wedge \ldots \wedge B_M \rightarrow B$. Thus $\vdash (A_1 \wedge \ldots \wedge A_N) \wedge (B_1 \wedge \ldots \wedge B_M) \rightarrow A \wedge B$. And since $\{A_1,\ldots,A_N\}$ and $\{B_1,\ldots,B_N\}$ are subsets of K their union is a subset of K and $K \vdash A \wedge B$.

<u>C[K]</u> is the set of all consequences of K ($C[K] = \{A \mid K \vdash A\}$). If $C[K]$ is the set of all wff, K is <u>contradictory</u>. If K is not contradictory K is <u>consistent</u>. The following theorem gives us a simple criterion to test whether a set of wff is consistent or contradictory.

THEOREM 6: K is contradictory if and only if there is a wff B such that $K \vdash B$ and $K \vdash \sim B$.

DEMONSTRATION: If K is contradictory $K \vdash B$ and $K \vdash \sim B$ whenever B is a wff. If $K \vdash B$ and $K \vdash \sim B$ then $K \vdash B \wedge \sim B$. Since $\vdash B \wedge \sim B \rightarrow A$ when A is any wff, $K \vdash A$.

THEOREM 7: If $\vdash A$ then $K \vdash A$ when K is any nonempty set of wff.

DEMONSTRATION: Let B be any wff in K. Utilizing Axioms 2 and 3, Modus Ponens and $\vdash A$, we see that $\vdash A \rightarrow A \vee \sim B$, $\vdash A \vee \sim B \rightarrow \sim B \vee A$ and hence $\vdash \sim B \vee A$. By the definition of $\rightarrow$ we see $\vdash B \rightarrow A$ and $K \vdash A$.

H. MAXIMAL CONSISTENT AND EXISTENCE-COMPLETE SETS

In this section we shall study two additional properties of sets of wff and we shall show if a set of wff has both these properties (ie. is both maximal consistent and existence-complete) then the set of wff has a model.

DEFINITION: A set of wff K is <u>maximal consistent</u> if and only if

1. K is consistent and
2. L is contradictory whenever $K \subset L$ and $K \neq L$.

THEOREM 8: If K is maximal consistent and $K \vdash A$ then $A \in K$.

DEMONSTRATION: Suppose for contradiction that $A \notin K$; then $K \cup \{A\}$ is contradictory. Hence $K \cup \{A\} \vdash \sim A$ or $K \vdash \sim A$ and K is contradictory. This contradiction establishes the theorem.

THEOREM 9: If $B \notin K$ then $\sim B \in K$, whenever K is maximal consistent.

DEMONSTRATION: Since $K \cup \{B\}$ is contradictory, $K \cup \{B\} \vdash \sim B$ or $K \vdash \sim B$. By theorem 8 $\sim B \in K$.

DEFINITION: A set of wff K is <u>existence-complete</u> if and only if whenever $\exists tA \in K$ there is an individual s such that $S^s_t A \in K$.

THEOREM 10: If K is maximal consistent and existence-complete, K possesses a model.

DEMONSTRATION: The model S is constructed as follows. The basic set of S consists of all the individuals in the predicate calculus. Each member of the basic set is displayed as term in S. For every predicate P of order n, include in S the n^{ary} relationship $R = \{(a_1,\ldots,a_n) \mid Pa_1\ldots a_n \in K\}$. K is defined in S by means of the identity map. Now we need to show that A holds in S if and only if $A \in K$. By construction all atomic wff $A \in K$ hold in S, and if a relation is in S it is because the corresponding atomic wff is in K.

Assume for contradiction that there is some composite wff A for which it is not true that A holds in S if and only if $A \in K$. Let C be the wff with the fewest connectives for which it is not true. Then there are three possible main connectives.

1. The main connective is ~.

Then there must be some B such that C is ~B. Since B has one less connective than C we know B holds in S if and only if $B \in K$.

If C holds in S then ~B holds in S and B does not hold in S, so B is not in K. By theorem 9, $\sim B \in K$ or $C \in K$.

Assume that ~B,(C), is in K. Since K is consistent B is not in K and hence does not hold in S (as B has fewer connectives than C). Since B does not hold ~B,(C), holds in S.

2. The main connective is $\vee$.

Then there must be wff D and E such that $C = D \vee E$. As D and E each have one less connective than C we know D holds in S if and only if $D \in K$ and E holds in S if and only if $E \in K$.

If C holds in S, $D \vee E$ holds in S and hence either D or E must hold in S. If D holds then $D \in K$. By theorem 7, since $\vdash D \to D \vee E$ (Axiom 2) $K \vdash D \to D \vee E$, so $K \vdash D \vee E$. Therefore by theorem 8, $D \vee E \in K$ or $C \in K$.

If $D \vee E \epsilon K$ $(C \epsilon K)$ we want to show that D or E (and hence $D \vee E$) holds in S. Assume for contradiction that neither D nor E hold in S. Then $D \notin K$ and $E \notin K$. By theorem 9 $\sim D \epsilon K$ and $\sim E \epsilon K$. Since $\vdash \sim D \wedge \sim E \rightarrow \sim D \wedge \sim E$ and $\{\sim D, \sim E\}$ is a subset of K, $K \vdash \sim D \wedge \sim E$. By theorem 8 $\sim D \wedge \sim E \epsilon K$. Then $\sim(\sim D \wedge \sim E) \notin K$. But $\sim(\sim D \wedge \sim E) \equiv D \vee E$ which is an element of K. This contradiction establishes that if $D \vee E \epsilon K$ then $D \vee E$ holds in S.

3. The main connective is $\forall$.

Then there is an individual x and a wff B such that $C = \forall x B$. As B has one less connective than C, $S^a_x B$ holds in S if and only if $S^a_x B \epsilon K$ (whenever a is an individual).

If $\forall x B$ holds in S, $S^a_x B$ holds in S whenever a is a member of the basic set of S. Therefore $S^a_x B \epsilon K$ whenever a is an individual of the calculus. If $\forall x B \notin K$ then by theorem 9 $\sim \forall x(B) \equiv \exists x(\sim B) \epsilon K$. Since K is existence-complete there is an individual a such that $S^a_x \sim B \epsilon K$, i.e. $\sim S^a_x B \epsilon K$. But if this were true K would be contradictory. Therefore $\forall x B \epsilon K$.

If $\forall x B \epsilon K$ then $S^a_x B \epsilon K$ when a is an individual. Therefore $S^a_x B$ holds in S whenever a is a member of the basic set of S and $\forall x B$ holds in S.

Therefore there cannot be a wff C with the fewest connectives such that it is not true that C holds in S if and only if $C \epsilon K$. This establishes the theorem.

I. THE COMPLETENESS OF THE PREDICATE CALCULUS

The following three statements are correct and any one of them establishes the completeness of the predicate calculus. The first is Gödel's Completeness Theorem and was demonstrated by him in 1930. The first statement, I, can be deduced from either II or III. II and III can each be deduced from the other. III is the extended completeness theorem and was first demonstrated by L. Henkin in 1949. [3]

I $\vdash A$ if and only if A is true.

II $K \vdash A$ if and only if $f(A)$ holds in S whenever S is a model of K under f such that A is defined in S under f.

III K is consistent if and only if K possesses a model.

THEOREM 11: If III then II.

DEMONSTRATION: Let S be a model of K under f such that A is defined in S under f. We need to show, given III, $K \vdash A$ if and only if $f(A)$ holds in S.

Assume for contradiction that $K \vdash A$ but $f(A)$ does not hold in S. Then $f(\sim A)$ holds in S. Then $K \cup \{\sim A\}$ has a model S and by III $K \cup \{\sim A\}$ is consistent. Since $K \vdash A$, $K \cup \{\sim A\} \vdash A$. But $K \cup \{\sim A\} \vdash \sim A$ also, so $K \cup \{\sim A\}$ is contradictory. This contradiction establishes that if $K \vdash A$ then A holds in S.

Now assume for contradiction that $f(A)$ holds in S but $K \vdash A$ is false. If $K \vdash A$ is false, A cannot be in K, so $K \cup \{\sim A\}$ must be consistent. By III $K \cup \{\sim A\}$ possesses a model S. There must be a mapping f such that, for every B in $K \cup \{\sim A\}$, $f(B)$ holds in S.

Then $f(\sim A)$ holds. This contradicts $f(A)$ holding in S. Thus if $f(A)$ holds in S, $K \vdash A$.

THEOREM 12: If II then III.

DEMONSTRATION: First we shall show that if K is consistent then K possesses a model. Assume for contradiction that K does not possess a model. Construct a structure S such that all the wff in K are defined in S under f. Since K has no model there must be wff A_i in K such that $f(A_i)$ does not hold in S. Let $K = K_1 \cup K_2$ where K_1 is the set of wff whose images hold and $K_2 = \{A_1, \ldots, A_n\}$ be the set of wff whose images do not hold. Since $f(A_i)$ does not hold, $f(\sim A_i)$ holds in S (S model of K_1 under f and A_i defined in S under f). Thus by II $K_1 \vdash \sim A_i$ and hence $K \vdash \sim A_i$. But since $A_i \epsilon K$, $K \vdash A_i$. This violates the assumption that K is consistent. The contradiction establishes that K has a model.

Next we shall show that if K possesses a model, K is consistent. Suppose for contradiction that K is contradictory. When S is a model of K under f and B is any wff defined in K under f, $K \vdash B$ and $K \vdash \sim B$. By II $f(B)$ and $f(\sim B)$ holds in S. But if $f(B)$ holds $f(\sim B)$ cannot, and vice versa. This contradiction establishes that K is consistent.

THEOREM 13: If II then I

DEMONSTRATION: We have already shown (in section F this chapter) that if A is provable A is true. It remains to show that, given II, if A is true then A is provable. Let $K = \{\sim A\}$ and S be a model of K under f. Since $\sim A$ is defined in S, A is defined in S.

And since A is true $f(A)$ must hold in S. By II, $K \vdash A$ $(K = \{\sim A\})$ and hence $\vdash \sim A \rightarrow A$. Thus by theorem 4, $\vdash A$.

From theorems 11 and 13 we see that if III holds then I, the completeness of predicate calculus, holds. The remaining part of this chapter will be a demonstration of III, the extended completeness theorem.

THEOREM 14: K is consistent if and only if K has a model.

DEMONSTRATION: First we will show that if K possesses a model K is consistent. Let S be any model of K under f and let B be a member of K. Then $f(B)$ holds. Assume for contradiction that K is contradictory then $K \vdash \sim B$ or there are $A_1, \ldots, A_n$ in K such that $\vdash A_1 \wedge \ldots \wedge A_n \rightarrow \sim B$. Since all provable wff are true, $A_1 \wedge \ldots \wedge A_n \rightarrow \sim B$ must be true. And since $A_1 \wedge \ldots \wedge A_n \rightarrow \sim B$ is defined in S, $f((A_1 \wedge \ldots \wedge A_n) \rightarrow \sim B)$ holds. This means $f(\sim B)$ holds whenever $f(A_1 \wedge \ldots \wedge A_n)$ holds (which is always since $A_1, \ldots, A_n$ are members of K). But if $f(B)$ holds, $f(\sim B)$ does not hold. This contradiction establishes that K must be consistent.

Next we must show that if K is consistent then K possesses a model. We must first extend the predicate calculus by adding individuals to it. Then we will construct a super set of K that is maximal consistent and existence-complete in the extended predicate calculus. From theorem 10 this super set of K possesses a model; hence K possesses a model.

Let $C_1, C_2, \ldots, C_n, \ldots$ be a sequence of predicate calculi. C_j is obtained from C_{j-1} by attaching $\{u_{j-1,k} | k$ is a natural number and $u_{j-1,k}$ are individuals not in C_{j-1}}. Let C_α be the predicate calculus whose predicates are the predicates of C_1 and whose individuals are the individuals of C_1 plus $\{u_{i,j} | i$ and j are natural numbers$\}$. Since the number of predicates in C_α is denumerable, the number of individuals in C_α is denumerable and the length of each wff in C_α is finite, the number of wff in C_α is denumerable. Thus we can let a particular enumeration of the wff in C_α be called the <u>standard ordering</u>.

LEMMA A: If K is consistent in C_1, K is consistent in C_α.

DEMONSTRATION: Assume for contradiction that K is contradictory in C_α. Then there is a finite subset of K, $\{A_1,\ldots,A_n\}$ such that $\vdash A_1\wedge\ldots\wedge A_n \rightarrow B\wedge \sim B$. There must be a proof of $A_1\wedge\ldots\wedge A_n \rightarrow B\wedge\sim B$ in C_α. Since K is consistent in C_1 there must not be a proof of $A_1\wedge\ldots\wedge A_n \rightarrow B\wedge\sim B$ in C_1. Then the proof must include individuals $x_1,\ldots,x_s$ in C_α that are not in C_1. Let $y_1,\ldots,y_s$ be individuals in C_1 not in the proof. Apply the transforms $I_{y_1}^{x_1},\ldots,I_{y_s}^{x_s}$ to the proof of $A_1\wedge\ldots\wedge A_n \rightarrow B\wedge\sim B$. By theorem 1 the result is a proof with individuals only in C_1. Thus K must be contradictory in C_1. This contradiction establishes lemma A.

Next we need to extend K to a set that is maximal consistent in C_1. Let $K_{1,1} = K$ and B_1 be the first wff of C_1 in the standard ordering such that $K_{1,1} \cup\{B_1\} = K_{1,2}$ is consistent. In general B_j is the first wff in the standard ordering of C_1 after B_{j-1} such that $K_{1,j} \cup \{B_j\} = K_{1,j+1}$ is consistent. Then $K_1 = \{A | A\in K_{1,n}, n$ is a natural number$\}$.

LEMMA B: K_1 is maximal consistent in C_1.

DEMONSTRATION: If K_1 is contradictory there is a finite subset of K_1 that is contradictory. Thus there is some natural number n such that $K_{1,n}$ is contradictory. Since this is impossible by construction, K_1 must be consistent. Next we need to show that K_1 is maximal consistent in C_1. Assume for contradiction that there is an A in C_1 such that $K_1 \cup \{A\}$ is consistent. Then A must have been included in one of the $K_{1,j}$'s and hence A is an element of K_1. Thus K_1 must be maximal consistent in C_1.

Now that we have a super set of K, K_1, that is maximal consistent in C_1 we shall extend K_1 to be existence-complete in C_2. Select the first wff in the standard ordering of K_1 that is in the form $\exists$tB. Let the first such wff be $\exists tD_1$. Add to K_1 the wff $S_t^{u_{1,1}}D_1$. Note that $K_1 \cup \{S_t^{u_{1,1}}D_1\}$ is consistent; for if it were contradictory then there would be a finite subset of K, $\{A_1,\ldots,A_n\}$, such that $\vdash S_t^{u_{1,1}}D_1 \wedge A_1 \wedge \ldots \wedge A_n \rightarrow B \wedge \sim B$ where $u_{1,1}$ is free in $S_t^{u_{1,1}}D_1$ and not occurring in $A_1 \wedge \ldots \wedge A_n \rightarrow B \wedge \sim B$. Thus $\vdash \exists tD_1 \dot{\rightarrow} A_1 \wedge \ldots \wedge A_n \rightarrow B \wedge \sim B$ or $K_1 \vdash B \wedge \sim B$. Since this would show K_1 to be contradictory and by lemma A, K_1 is consistent, $K_1 \cup \{S_t^{u_{1,1}}D\}$ must be consistent. Wff are added to K_1 in this manner until there are no more wff in the form $\exists tD_j$ in K_1 without corresponding wff $S_t^{u_{1,j}}D_j$ in the extension of K_1. For instance if $\exists tD_j$ is the j^{th} wff of the specified form in K_1, then $S_t^{u_{1,j}}D_j$ is added to $K_1 \cup \{S_t^{u_{1,1}}D_1,\ldots,S_t^{u_{1,j-1}}D_{j-1}\}$. Once K_1 has been extended to be existence-complete in C_2 the resulting set is extended to be maximal consistent in C_2. This set is denoted by K_2.

K_3 is constructed in a similar manner. K_2 is a subset of K_3. K_3 is maximal consistent in C_3 and $S^a_t D \in K_3$ for some individual a of C_3 whenever $\exists tD \in K_2$.

This procedure is continued. Each time the set K_n which is maximal consistent in C_n is extended in two steps to a set K_{n+1} which is maximal consistent in C_{n+1}. First for wff of the form $\exists tD$ in K_n, the wff $S^a_t D$ (where a is an individual in C_{n+1}) is added to K_n. Secondly, this resultant set is extended to a maximal consistent set in C_{n+1}.

Finally let $K_\alpha = \{A | A \in K_n$, n is a natural number$\}$. We wish to show that K_α is maximal consistent and existence-complete.

LEMMA C: K_α is maximal consistent in C_α.

DEMONSTRATION: Assume for contradiction that K_α is contradictory. Then there is a finite subset of K_α, $\{A_1, \ldots, A_n\}$ such that $\vdash A_1 \wedge \ldots \wedge A_n \rightarrow B \wedge \sim B$ in C_α. Then there is a natural number m such that $\{A_1, \ldots, A_n\}$ is a subset of K_m. Thus $\vdash A_1 \wedge \ldots \wedge A_n \rightarrow B \wedge \sim B$ in C_m and K_m is contradictory. This contradiction demonstrates that K_α must be consistent in C_α. Let A be any wff in C_α such that $K_\alpha \cup \{A\}$ is consistent in C . Now if $A \in C_\alpha$ then there is a natural number n such that $A \in C_n$. But if $K_\alpha \cup \{A\}$ is consistent then $K_{n-1} \cup \{A\}$ is consistent. This means $A \in K_n$ and so $A \in K_\alpha$. Thus K_α is maximal consistent.

LEMMA D: K_α is existence-complete in C_α.

DEMONSTRATION: Suppose $\exists tD \in K$. Then there is a natural number m such that $\exists tD \in K_m$. Then by construction there is an individual a in

C_{m+1} such that $S^a_t D \epsilon K_{m+1}$. Therefore $S^a_t D \epsilon K_\alpha$ and K_α is existence-complete in C_α . [3]

Thus K_α possesses a model (theorem 10). But K is a subset of K_α , so K possesses a model. Thus we have shown that K is consistent if and only if K possesses model (III). Earlier we showed that if III then I, a wff is true if and only if it is provable. The predicate calculus is complete.

Theorem 14 was originally proven by Gödel in 1930. The proof given here is due to Henkin as simplified by Hasenjaeger in 1953. Other proofs have been published by Rasiowa-Sikorski using algebraic (Boolean) methods and by Beth using topological methods. Still other proofs have been given by Beth and Hintikka.

SUMMARY

This paper has shown two examples of a formalized deduction theory. The value of these systems rests with their precision. The proofs in each are based strictly on the structure; no subjective evaluation is required or permitted. In addition, each of these mathematical disciplines was shown to be complete. Each expression in the appropriate system can be shown either to be provable or not provable. This concept cannot be fully appreciated until we see that even a discipline as basic as arithmetic cannot make this same claim of completeness.

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